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ENTROPIC INEQUALITIES FOR TWO COUPLED SUPERCONDUCTING CIRCUITS

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Abstract

We discuss the known construction of two interacting superconducting circuits based on Josephson junctions, which can be precisely engineered and easily controlled. In particular, we use the parametric excitation of two circuits realized by an instant change of the qubit coupling to study entropic and information properties of the density matrix of a composite system. We obtain the density matrix from the initial thermal state and perform its analysis in the approximation of small perturbation parameter and sufficiently low temperature. We also check the subadditivity condition for this system both for the von Neumann entropy and deformed entropies and check the dependence of mutual information on the system temperature. Finally, we discuss the applicability of this approach to describe the two coupled superconducting qubits as harmonic oscillators with limited Hilbert space.

Keywords: entropic inequalities, subadditivity condition, composite system, superconducting circuit.

1. Introduction

The idea of using the Josephson junction to engineer superconducting circuits, where current and voltage are considered as analogs of the position and momentum in a parametric oscillator [1], is now employed to study the properties of qubits associated with the state of such circuits. The nonstationary quantum states of current and voltage in such circuits have been extensively studied [2–5] for more than two decades, and enormous progress in experimental realization [6, 7] of these systems has been made, which gave rise to a whole new area of research called quantum information processing. In the recent works [8–11], the information properties of qubits are studied as the resource for future quantum technologies. The particular properties of the multiqubit states to be analyzed are entropic and information inequalities [12, 13], which serve as the basis for quantum information processing.

In this work, we aim at studying the subadditivity condition for von Neumann entropy [14] and q-entropies [15] of the bipartite quantum system [16–22], on a specific example of two coupled superconducting circuits [23]. To analyze entropic and information properties of the composite system, we use the parametric excitation of two circuits realized by an instant change of the qubit coupling. Initially, we assume the system to be in the thermal state with a corresponding thermal density matrix, which is

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then analyzed in the approximation of small perturbation parameter and sufficiently low temperature. We also consider the dependence of mutual information on temperature of the system and discuss the applicability of our approach to describe the two coupled superconducting qubits as harmonic oscillators with limited Hilbert space.

2. Theoretical Model

We start with a simple model of two interacting harmonic oscillators in the thermal state. As we would like to consider only the first two levels in each of the circuits, we are working in the limit $T \to 0$. We discuss this approximation and its applicability more thoroughly in Sec. 6.

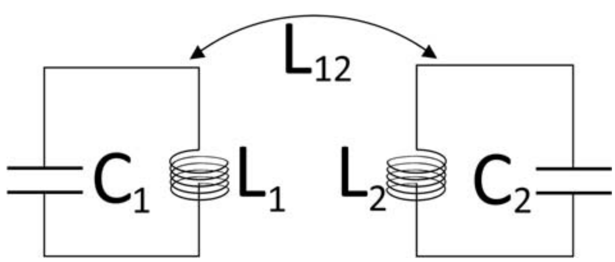

Fig. 1. Schematic representation of two superconducting circuits modeled as harmonic oscillators coupled by a mutual inductance.

The Hamiltonian of the system of two coupled resonant circuits shown in Fig. 1 can be written in the following simple form:

$$\hat{H} = \hat{H}_1 + \hat{H}_2 + \hat{V}, \tag{1}$$

where first two terms correspond to two independent LC circuits, and the third term defines their coupling

$$\hat{H}_i = \frac{L_i \hat{I}_i^2}{2} + \frac{\hat{Q}_i^2}{2C_i}, \qquad \hat{V} = L_{12}\hat{I}_1\hat{I}_2, \quad i = 1,2.$$

For convenience and due to the duality between mechanical oscillators and LC-circuits, we introduce canonical position and momentum operators, which we will be using hereinafter, through the following change of variables [24]:

$$\hat{x}_j = -L_j C^{1/2}\hat{I}_j, \qquad \hat{p}_j = C_j^{-1/2}\hat{Q}_j, \qquad [\hat{x}_j, \hat{p}_k] = i\hbar\delta_{jk}, \quad j,k = 1,2.$$

Using this substitution, we can rewrite Eq. (1) as

$$\hat{H} = \frac{\hat{p}_1^2}{2m} + \frac{m\omega_1^2\hat{x}_1^2}{2} + \frac{\hat{p}_2^2}{2m} + \frac{m\omega_2^2\hat{x}_2^2}{2} + gm\hat{x}_1\hat{x}_2\omega_1\omega_2, \tag{2}$$

where $g = L_{12}(L_1L_2)^{-1/2}$ is the qubit coupling constant, and $\omega_{1,2}$ are the eigenfrequencies of the circuits. For simplicity, here and later we assume that $m = 1$, $\omega_1 = 1$, and $\omega_2 = \lambda$.

Next, we get rid of the cross term and diagonalize the Hamiltonian [25]. For this, we apply the following rotation by an angle ϕ, using the theorem that a quadratic form is reduced to a diagonal form by an orthogonal transform

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} x_1' \\ x_2' \end{pmatrix}. \tag{3}$$

After this, we arrive at the Hamiltonian

$$\hat{H} = \frac{\hat{p}_1^2}{2} + \frac{\hat{p}_2^2}{2} + \frac{\cos^2\phi\hat{x}_1' + \sin^2\phi\hat{x}_2'^2}{2} + \cos\phi\sin\phi\hat{x}_2'\hat{x}_1' + \frac{\lambda^2(\sin^2\phi\hat{x}_1' + \cos^2\phi\hat{x}_2')}{2}$$
$$-\lambda^2\sin\phi\cos\phi\hat{x}_2'\hat{x}_1' + g\lambda(-\cos\phi\sin\phi\hat{x}_1'^2 + \cos\phi\sin\phi\hat{x}_2'^2) + g\lambda(\cos^2\phi - \sin^2\phi)\hat{x}_2'\hat{x}_1'. \tag{4}$$

Equating to zero the cross terms in Eq. (4), we obtain a system of two independent harmonic oscillators

$$\hat{H} = \frac{\hat{p'}_1^2}{2} + \frac{\Omega_1^2 \hat{x'}_1^2}{2} + \frac{\hat{p'}_2^2}{2} + \frac{\Omega_2^2 \hat{x'}_2^2}{2}, \tag{5}$$

where the new resonant frequencies and the rotation angle in the approximation of small angles are

$$\Omega_1^2 \approx 1 - 2g\lambda\phi + \lambda^2\phi^2, \qquad \Omega_2^2 \approx \phi^2 + \lambda^2 + 2g\lambda\phi, \qquad \phi \approx g\lambda/(\lambda^2 - 1)\ . \tag{6}$$

3. Density Matrix

As the oscillators under study are in the thermal state in the low-temperature limit, the density matrix of the system reduced to the 4×4 subspace reads

$$\rho = \frac{1}{Z(T)} \begin{pmatrix} \exp^{-E_{00}/T} & 0 & 0 & 0 \\ 0 & \exp^{-E_{01}/T} & 0 & 0 \\ 0 & 0 & \exp^{-E_{10}/T} & 0 \\ 0 & 0 & 0 & \exp^{-E_{11}/T} \end{pmatrix}, \tag{7}$$

where $E_{nm} = \Omega_1(n+1/2) + \Omega_2(m+1/2)$ and $Z(T) = Z_1(T)Z_2(T) = [4\sinh(\omega_1/2T)\sinh(\omega_2/2T)]^{-1}$. In order to return to the initial system with two coupled resonant circuits, we should perform the transform, which decomposes the old eigenstates in the new rotated basis

$$\tilde{\rho} = U^{-1}\rho\, U. \tag{8}$$

In the basis of the harmonic oscillators eigenstates, we denote the coefficients of the transform $U_{nmn'm'}$ and calculate them using the eigenfunctions of the harmonic oscillators,

$$U_{nmn'm'} = \frac{1}{\pi}\int_{-\infty}^{\infty} \exp\left[-\frac{x_1^2}{2} - \frac{x_2^2}{2l^2} - \frac{x_1'^2}{2L_1^2} - \frac{x_2'^2}{2L_2^2}\right] \frac{dx_1\, dx_2}{\sqrt{lL_1L_2}\sqrt{2^n 2^m n! m!}\sqrt{2^{n'}2^{m'}n'!m'!}} \times H_n(x_1)\, H_m(x_2/l)\, H_{n'}\left(x_1'/L_1\right) H_{m'}\left(x_2'/L_2\right), \tag{9}$$

where

$$l = \sqrt{\hbar/m\omega} = \sqrt{1/\lambda}, \qquad \omega = \lambda, \qquad L_1 = \sqrt{1/\Omega_1}, \qquad L_2 = \sqrt{1/\Omega_2},$$

and H_i are the corresponding Hermite polynomials.

After calculating the matrix elements of the transform $U_{nmn'm'}$ (see Appendix for the details), we obtain the final density matrix to work with from Eq. (8), which is shown in Fig. 2 at various temperatures of the system. One can see that the approximation used is valid only for very low temperatures, below 100 mK, which is the standard one for superconducting qubits. At higher temperatures, the populations of the higher energy levels become non-negligible, thus, demanding to take into account a larger subspace than the 4×4 matrix we are discussing here.

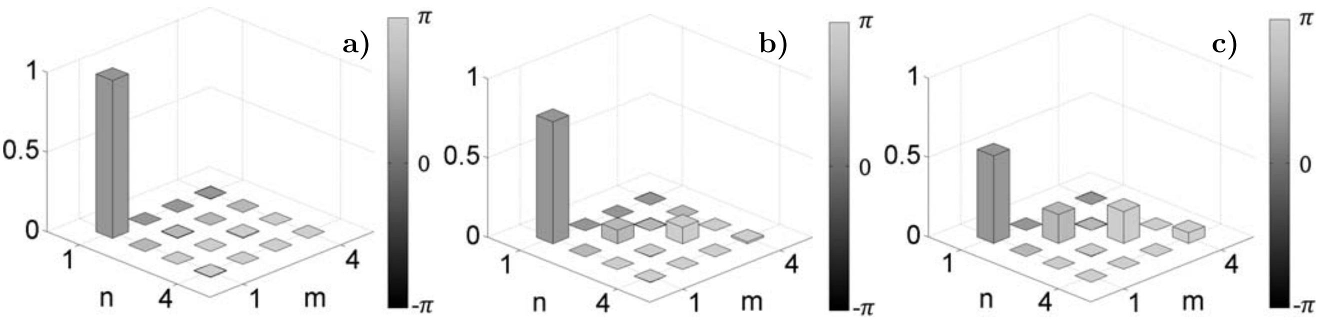

Fig. 2. Density matrices calculated from Eq. (8) at different temperatures $T = 100$ mK (a), $T = 250$ mK (b), and $T = 500$ mK (c). Here, the off-diagonal elements are nonzero but not visible due to the scale difference.

4. Calculation of Entropies

Using the density matrix obtained, we can calculate the entropic properties of the system. The main properties we are interested in are entropies of the bipartite system and its subsystems (single qubits) and the mutual information, which can be obtained from the subadditivity condition.

The two types of entropies we are discussing here are the von Neumann entropy $S = -\mathrm{Tr}\,\tilde{\rho}\ln\tilde{\rho}$ and the Tsallis entropy, which is equal to the von Neumann entropy in the limit of $q \to 1$,

$$S_q^T = -\mathrm{Tr}\,\tilde{\rho}\ln_q\tilde{\rho}, \tag{10}$$

where the q logarithm is defined as

$$\ln_{q>0}\rho = \begin{cases} \dfrac{\rho^{q-1} - \hat{I}}{1-q}, & q \neq 1, \\ \ln\rho, & q = 1. \end{cases}$$

Since the matrix elements of the density matrix depend on the system temperature, we calculate Eq. (10) for various temperatures and show the results in Fig. 3. The temperature here and later is given in units of frequency with respect to the eigenfrequency of qubits ω.

One can see that the q entropies in Fig. 3 are collected in such a way that the entropies with $q < 1$ are located higher than the von Neumann entropy ($q = 1$), while the entropies with $q > 1$ are located under von Neumann entropy. We explain this behavior as follows. The higher the q value, the higher the influence of the biggest terms in the distribution on entropy, and the more deterministic the system behavior. As the main elements of the density matrix are the four diagonal ones, higher q values lead to lower value of q entropy for the density matrix.

5. Verifying the Subadditivity Condition

To check the subadditivity condition for the bipartite system consisting of two resonant circuits, we divide the calculated density matrix from Eq. (8) into the density matrices of the subsystems as follows:

$$\tilde{\rho}_1 = \begin{pmatrix} \rho_{11}+\rho_{22} & \rho_{13}+\rho_{24} \\ \rho_{31}+\rho_{42} & \rho_{33}+\rho_{44} \end{pmatrix}, \qquad \tilde{\rho}_2 = \begin{pmatrix} \rho_{11}+\rho_{33} & \rho_{12}+\rho_{34} \\ \rho_{21}+\rho_{43} & \rho_{22}+\rho_{44} \end{pmatrix}. \tag{11}$$

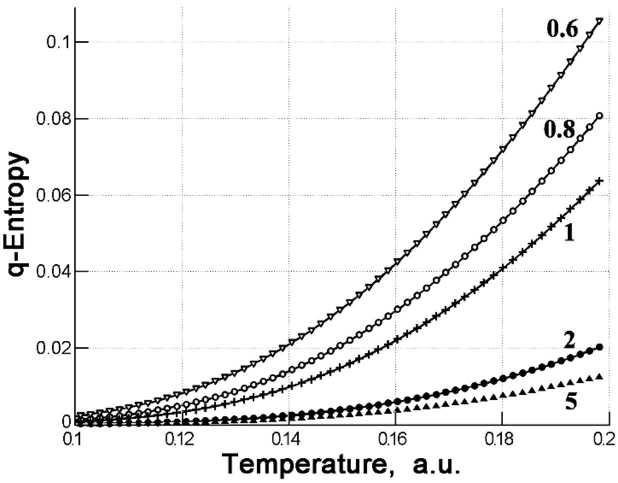

Fig. 3. Dependences of q entropies on temperature calculated from Eq. (10). Figures near the curves correspond to the value of q, and the middle curve is the von Neumann entropy.

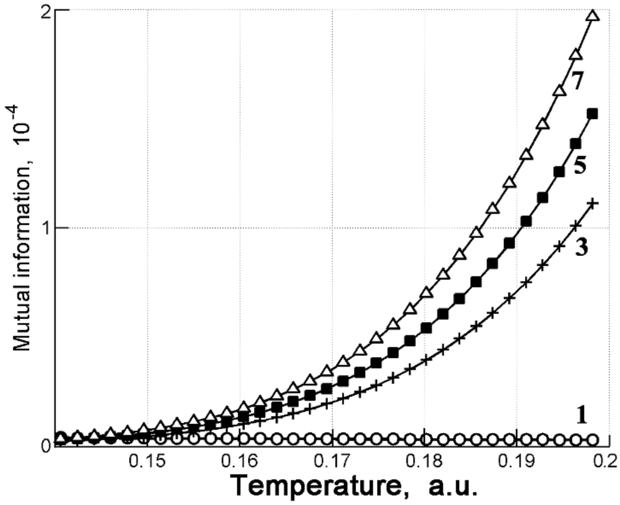

Fig. 4. Mutual information versus temperature. Figures near the curves correspond to the value of q.

The subadditivity condition for the corresponding q entropies reads

$$S_q^T(\tilde{\rho}) \leq S_q^T(\tilde{\rho}_1) + S_q^T(\tilde{\rho}_2), \tag{12}$$

and the mutual information is

$$I = -S_q^T(\tilde{\rho}) + S_q^T(\tilde{\rho}_1) + S_q^T(\tilde{\rho}_2) \geq 0. \tag{13}$$

Using Eq. (13) we calculate the values of mutual information for various temperatures of the system; see Fig. 4. The fact that the mutual information is positive in the whole temperature range demonstrates the validness of the subadditivity condition for the system.

6. Applicability of the Approximation

In this section, we discuss the applicability of the used approach to approximate the system of two harmonic oscillators by two qubits in the limit of low temperature. We calculate the purity parameter $\mu = \mathrm{Tr}\,\rho^2$, first for the reduced density 4×4 matrix and then for the remaining part of the density matrix; we present the result in Fig. 5 versus the system temperature. Also we present the sum of nondiagonal elements of the reduced 4×4 matrix to show the region where it is larger that the purity error.

In addition, in Fig. 5 we plot the purity of the density 4×4 matrix versus temperature to show the applicability of the approach used. One can see that the 4×4 subspace approximation is valid below $T \approx 0.2$, where $\mu \approx 1$, i.e., the approach used is valid for temperatures up to 100 mK (or 0.2 in relative units). So, for all the calculations with the reduced density 4×4 matrix, we normalize its elements to obtain $\mu = 1$.

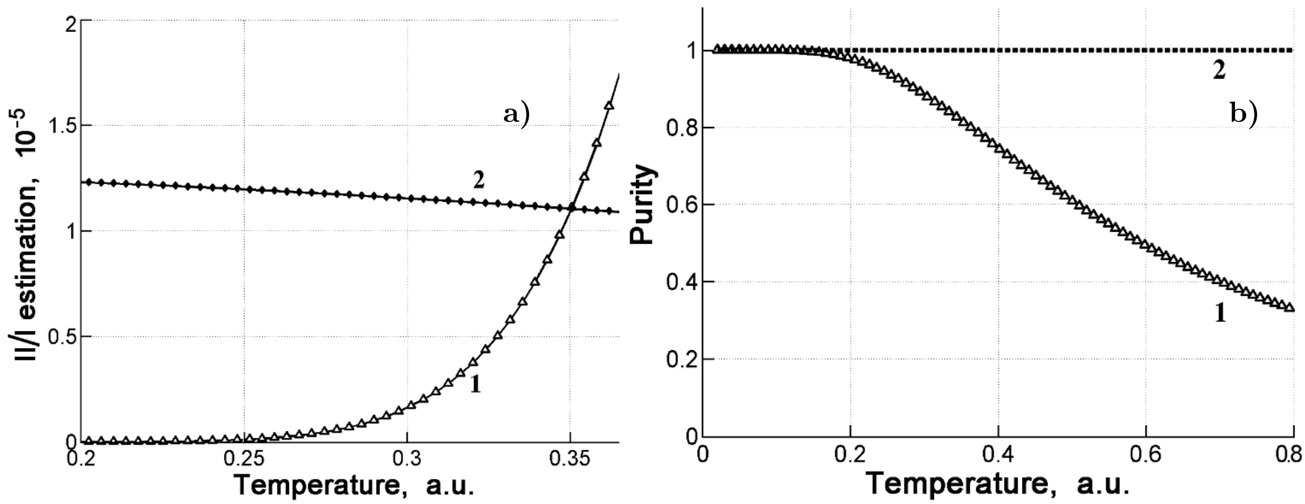

Fig. 5. Approximation applicability in terms of temperature: The inverse relation (a) of the purity of the 4-level approximation to the purity of the remaining part of the density matrix (curve 1) and the sum of nondiagonal matrix elements squared (curve 2). The dependence of the purity of the 4-level approximation (b) on temperature (curve 1) and the ideal case of $\mu = 1$ (curve 2).

7. Summary and Conclusions

We performed the analysis of a system of two coupled superconducting circuits modeled by two interacting harmonic oscillators in the low-temperature limit and derived the system density matrix in the small-perturbation approximation. Further we checked that the calculated density matrix of the bipartite system satisfies the entropic inequalities for the von Neumann entropy and the Tsallis entropy and considered the dependence of the mutual information on the system temperature. Finally, we evaluated the purity parameter of the system and verified the applicability of the elaborated approach to describe a system of two coupled superconducting qubits as harmonic oscillators with limited Hilbert space.

In a future paper, we will compare our findings with experimental data and generalize the elaborated approach for an arbitrary rotation angle ϕ and for larger number of qubits.

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Appendix

Here, we describe in detail how the matrix elements are calculated.

In the basis of the eigenstates $|m,n\rangle$, we denote the coefficients of decomposition $U_{nmn'm'}$ and calculate the matrix elements of the $\hat{U}$ in the eigenvalues basis of the Hamiltonian (Fock basis).

For the transition from the old basis to a new one $|e_k\rangle \to |\tilde{e_k}\rangle$, we use the following decomposition:

$$|\tilde{e_k}\rangle = \sum_m U_{km} |e_m\rangle ; \qquad U_{kn} = \int_{-\infty}^{\infty} \Psi_n^*(x)\tilde{\Psi}_k(x)\, dx. \tag{14}$$

So, for the density matrix we obtain

$$\rho_{nmn'm'} = \langle nm|\, \hat{\rho}\, |n'm'\rangle = \frac{1}{Z(T)} \langle nm|\, e^{-\beta \hat{H}}\, |n'm'\rangle . \tag{15}$$

In view of the harmonic oscillator eigenfunctions $\Psi_n(x) = \dfrac{1}{\sqrt{2^n n! \sqrt{\pi}}} e^{-x^2/2}\, \hat{H}_n(x)$, where $\omega = 1$, the unitary transform matrix has the integral form

$$\begin{aligned} U_{nmn'm'} &= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{dx_1}{\sqrt{2^n 2^m n! m!}} \frac{dx_2}{\sqrt{2^{n'} 2^{m'} n'! m'!}} \exp\left(-\frac{x_1^2}{2} - \frac{x_2^2}{2l^2}\right) \exp\left(-\frac{\Omega_1 x_1'^2}{2} - \frac{\Omega_2 x_2'^2}{2}\right) \\ & \times \frac{1}{\sqrt{l}} \frac{1}{\sqrt{L_1 L_2}} H_n(x_1)\, H_m\left(\frac{x_2}{l}\right) H_{n'}\left(\frac{x_1'}{L_1}\right) H_{m'}\left(\frac{x_2'}{L_2}\right). \end{aligned} \tag{16}$$

After some algebra, we arrive at

$$\begin{aligned} U_{nmn'm'} &= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{dx_1\, dx_2}{\sqrt{l}\sqrt{L_1 L_2}} \exp\left(-\frac{x_1^2}{2} - \frac{x_2^2}{2l^2}\right) \frac{H_n(x_1)}{\sqrt{2^{n'} 2^{m'} n'! m'!}} \frac{H_m(x_2/l)}{\sqrt{2^n 2^m n! m!}} \\ & \times \exp\left[-\frac{(\cos\phi x_1 + \sin\phi x_2)^2}{2} (\lambda^2 \sin^2\phi + \cos^2\phi - 2g\lambda \sin\phi\cos\phi) \right. \\ & \left. - \frac{(-\sin\phi x_1 + \cos\phi x_2)^2}{2} (\sin^2\phi + \lambda^2\cos^2\phi + 2g\lambda\sin\phi\cos\phi)\right] \\ & \times H_{n'}\left(\frac{\cos\phi x_1 + \sin\phi x_2}{L_1}\right) H_{m'}\left(\frac{-\sin\phi x_1 + \cos\phi x_2}{L_2}\right), \end{aligned} \tag{17}$$

where we use the notation $\quad l = \sqrt{\hbar/m\omega} = \sqrt{1/\lambda}, \quad \omega = \lambda, \quad L_1 = \sqrt{1/\Omega_1}, \quad L_2 = \sqrt{1/\Omega_2},$

$$\Omega_1^2 = \lambda^2 \sin^2\phi + \cos^2\phi - 2g\lambda\sin\phi\cos\phi \quad \text{and} \quad \Omega_2^2 = \sin^2\phi + \lambda^2\cos^2\phi + 2g\lambda\sin\phi\cos\phi.$$

In the approximation of small angles ($\cos\phi \approx 1$ and $\sin\phi \approx \phi$), the last terms read

$$\begin{aligned} & H_{n'}\left(\frac{\cos\phi x_1 + \sin\phi x_2}{L_1}\right) H_{m'}\left(\frac{-\sin\phi x_1 + \cos\phi x_2}{L_2}\right) \approx H_{n'}\left(\frac{x_1 + \phi x_2}{L_1}\right) H_{m'}\left(\frac{x_2 - \phi x_1}{L_2}\right), \\ & \Omega_1^2 \approx \lambda^2\phi^2 + 1 - 2g\lambda\phi, \qquad \Omega_2^2 \approx \phi^2 + \lambda^2 + 2g\lambda\phi, \end{aligned}$$

where we denote $K = \sqrt{l L_1 L_2} \approx (\lambda^2(\lambda^2\phi^2 + 1 - 2g\lambda\phi)(\phi^2 + \lambda^2 + 2g\lambda\phi))^{-1/8}$ and use $H_0(x) = 1$ and $H_1(x) = 2x$. With these simplifications, we obtain the following expression for the matrix elements:

$$\begin{aligned} U_{nmn'm'} &= \frac{1}{\pi K} \int_{-\infty}^{\infty} \exp\left[-\left(\frac{1}{2}(\Omega_1 + \phi^2\Omega_2 + 1)x_1^2 + (\phi\Omega_1 - \phi\Omega_2)x_1 x_2 + \frac{1}{2}(\phi^2\Omega_1 + \Omega_2 + \lambda)x_2^2\right)\right] \\ & \times H_n(x_1)\, H_m\left(\frac{x_2}{l}\right) H_{n'}\left(\frac{x_1 + \phi x_2}{L_1}\right) H_{m'}\left(\frac{x_2 - \phi x_1}{L_2}\right) \frac{dx_1}{\sqrt{2^n 2^m n! m!}} \frac{dx_2}{\sqrt{2^{n'} 2^{m'} n'! m'!}}. \end{aligned} \tag{18}$$

Using this equation, we calculate the matrix elements of the transform and obtain the density matrix from Eq. (8).

One can find more detailed calculations in [26].

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