Collective quantum coherent oscillations in a globally coupled array of superconducting qubits

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We report a theoretical study of coherent collective quantum dynamic effects in an array of N qubits (two-level systems) incorporated into a low-dissipation resonant cavity. Individual qubits are characterized by energy level differences Δ_i and a spread of Δ_i is taken into account. Noninteracting qubits display coherent quantum beatings with N different frequencies, i.e., $\omega_i = \Delta_i/\hbar$. Virtual emission and absorption of cavity photons provides a long-range interaction between qubits. In the presence of such interaction we analyze quantum correlation functions of individual qubits $C_i(t)$ to obtain *two collective quantum-mechanical coherent oscillations*, characterized by frequencies $\omega_1 = \bar{\Delta}/\hbar$ and $\omega_2 = \tilde{\omega}_R$, where $\tilde{\omega}_R$ is the resonant frequency of the cavity renormalized by interaction. The amplitude of these oscillations can be strongly enhanced in the resonant case when $\omega_1 \simeq \omega_2$. These collective quantum oscillations can be directly observed, e.g., by measurements of frequency dependent transmission coefficient $D(\omega)$ of electromagnetic field propagating in a transmission line coupled to the system.

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I. INTRODUCTION

Great attention is devoted to theoretical and experimental studies of various superconducting qubits [1–3]. It can be small and large Josephson junctions (charge and phase qubits), rf superconducting quantum interference devices (SQUIDs), and many-junction superconducting quantum interferometers (flux qubits), just to name a few. A crucial property of such systems is that at low temperatures they can be modeled as quantum-mechanical two-state systems displaying coherent quantum dynamical phenomena, i.e., quantum beating between two states [4–7], and, in the presence of externally applied radiation, microwave induced Rabi oscillations, Ramsey fringes, etc., [8–10]. For single qubits these effects have been analyzed theoretically and observed experimentally.

As we turn to diverse systems containing many interacting qubits quantum dynamics becomes more complex and interesting. First of all due to a spread of parameters of individual qubits they perform quantum beating oscillations with different frequencies equal to (in the noninteracting case) $\omega_i = \Delta_i/\hbar$, where Δ_i is the energy level splitting of a single qubit. For example, in Ref. [11] a system of seven flux qubits, i.e., three-junction superconducting quantum interferometers, has been studied to reveal a behavior corresponding to the presence of seven different two-level systems. Thus, the presence of an unavoidable spread of parameters of qubits results in a nonsynchronized quantum dynamics of noninteracting qubits. Similar results have also been obtained for a single Josephson junction containing a large amount of microscopic two-level systems randomly distributed in its insulator interlayer [12–14]. Therefore, one could ask: Is it possible to observe *collective quantum coherent phenomena* arising in the whole system?

In order to obtain such synchronized behavior in systems of many qubits an interaction between them has to be provided. One way to do so is through absorption and emission of virtual photons in a resonator. Coupling between a single qubit and a low-dissipation superconducting resonator was theoretically studied in [15], and it has been experimentally realized in numerous works (see, e.g., [16–18]). Incorporating many qubits in a resonator provides a long-range interaction between them [15,19–22]; such a setup has been used in Refs. [11,23,24].

A convenient method to observe coherent quantum phenomena is to measure the frequency dependent transmission (reflection) coefficient of electromagnetic field propagating in a transmission line [25] coupled to the system showing coherent quantum-mechanical behavior. This so-called dispersive readout of qubits has been used to study the diverse quantum phenomena in a resonator coupled to a single [16,26] or many qubits [11,23,24].

In this paper we show that in the presence of long-range interaction an array of N qubits displays, beyond the quantum beating oscillations on individual frequencies ω_i, *two collective coherent quantum oscillations*. These quantum oscillations are characterized by two frequencies, $\omega_1 = \bar{\Delta}/\hbar$ and $\omega_2 = \tilde{\omega}_R$, where $\bar{\Delta}$ is the energy level difference averaged over an ensemble of qubits, and $\tilde{\omega}_R$ is the resonator frequency renormalized by interaction. Moreover, we obtain that the amplitude of these oscillations can be strongly enhanced in the resonant case as $\omega_1 \simeq \omega_2$.

The usual borrowed from quantum optics theoretical framework for analyzing a system of qubits interacting with cavity modes is the Tavis-Cummings/Dicke model [27–29], which initially assumed identical qubits. This model allows one to analyze various interesting quantum phenomena, e.g., the vacuum Rabi splitting, a few photons induced Rabi splitting, etc. One of the main results of this theory is the nonlinear ($\sim\sqrt{N}$) enhancement of coupling between qubits and resonator mode which has been observed experimentally [30,31]. Extensions of this model to systems with parameter spread have also been considered [32]. However, the Tavis-Cummings model relies on two important assumptions: (1) rotating wave approximation, applicable only if qubit level splittings approximately coincide with the resonator frequency and that (2) the splittings themselves are not influenced by the emergent interaction. One can imagine that (2) can be particularly restrictive for obtaining a collective coherent behavior.

In our analysis we circumvent these limitations while reproducing Tavis-Cummings results in a limiting case. Thus, we use an alternative theoretical approach borrowed from mesoscopic physics [33–36]. In this approach the qubits and resonant modes are characterized by continuous dynamic variables, and by tracing out the resonant mode degree of freedom we obtain an effective long-range interaction between qubits. Next, by making use of the instanton approach the energy level splittings and the time-dependent correlation function of interacting qubits, $C_i(t)$, will be determined self-consistently. The quantum-mechanical dynamics of interacting qubits will be characterized quantitatively by the time-dependent correlation function of individual qubits, i.e., $C_i(t)$. We show that the Fourier transform of this correlation function, $C_i(\omega)$, can be directly measured through the frequency dependent resonant drops of transmission coefficient $D(\omega)$ of electromagnetic field propagating in the transmission line coupled to a system. Therefore, obtained collective quantum oscillations result in additional resonant drops in the dependence of $D(\omega)$.

The paper is organized as follows: In Sec. II we derive the effective Lagrangian for a multiqubit system coupled to a resonator and obtain corrections to qubit level splittings. In Sec. III correlation functions for qubits are obtained. In Sec. IV we discuss possible signatures of collective quantum coherent oscillations in a transmission line experiment. Section V provides conclusions.

II. EFFECTIVE INTERACTION OF QUBITS COUPLED TO A RESONATOR

We study the collective coherent quantum phenomena for a particular example of an array of N rf SQUIDs inductively coupled to a resonant cavity. Each rf SQUID (single qubit) is characterized by a dynamic variable—Josephson phase $\varphi_i(t)$. Potential relief for the Josephson phase $U(\varphi_i)$ can be tuned by externally applied magnetic field to have a double-well form. The resonator is characterized by two parameters L_0 and C_0, the inductance and capacitance per unit length, accordingly. The resonator frequencies are written as $\omega_R = ck_n$, where $c = 1/\sqrt{L_0C_0}$ and $k_n = \pi n/\ell$, where ℓ is the size of the resonator, $n = 1,2\ldots$. As the resonator has an extremely high quality factor only one wave vector will be important in the dynamics of coupled qubits and photons of the resonator. Mutual inductance M provides an interaction between rf SQUIDs and the resonator. The schematic of such a system is presented in Fig. 1. The classical nonlinear dynamics of such system has been studied, e.g., in Ref. [37].

We start our quantitative analysis with the partition function Z written as a path integral over Josephson phases $\varphi_i(\tau)$, and the charge variable characterizing photon states in the resonator $Q(\tau)$, where τ is the imaginary time, i.e.,

$$Z = \int D[\varphi_i,Q]\exp\{-S[\varphi_i,Q]/\hbar\}, \qquad (1)$$

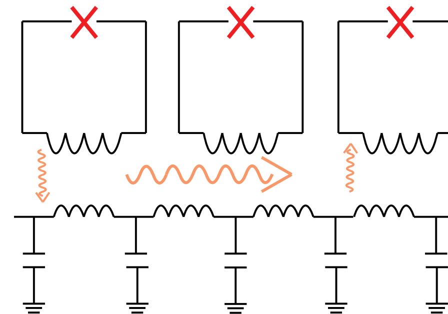

FIG. 1. (Color online) The schematic of an array of rf SQUIDs incorporated into a resonator. An interaction through emission (absorption) of virtual photons is shown.

where the action $S[\varphi_i,Q]$ is

$$\begin{aligned} S[\varphi_i,Q] &= \int_0^{\hbar/(k_BT)} d\tau[L_{\text{qubits}} + L_{\text{res}} + L_{\text{int}}],\\ L_{\text{qubits}} &= E_J\sum_i\left\{\frac{(\dot\varphi_i)^2}{2\omega_p^2} - \frac{\alpha_i\varphi_i^2}{2} + \frac{\varphi_i^4}{24}\right\},\\ L_{\text{res}} &= \frac{m}{2}[\dot Q^2 + \omega_R^2Q^2], \quad m = L_0\ell/2,\\ L_{\text{int}} &= iE_JQ\sum_i\eta_i\dot\varphi_i. \end{aligned} \qquad (2)$$

Here, E_J and ω_p are the Josephson coupling energy and the plasma frequency, accordingly. The parameters η_i and α_i can be expressed through the parameters of ith rf SQUID and its mutual inductance with the resonator. We consider the quantum dynamics involving the two low-lying levels of each qubit only and therefore the real potential $U(\varphi_i)$ of rf SQUIDs is truncated to the model potential of φ^2-φ^4 form. For our particular case of rf SQUIDs incorporated into a low-dissipation resonator the parameters have been obtained explicitly in Ref. [38]. Next, we trace out [22,33] the partition function over the charge variable $Q(\tau)$ and obtain the effective action S_{eff} of N globally coupled two-level systems:

$$\begin{aligned} S_{\text{eff}} &= E_J\sum_{i=1}^N\int_0^{\hbar/(k_BT)}d\tau\left[\frac{(\dot\varphi_i)^2}{2\omega_p^2} - \frac{\alpha_i\varphi_i^2}{2} + \frac{\varphi_i^4}{24}\right]\\ &\quad + \frac{E_J}{2}\sum_{i,j}\xi_i\xi_j\int_0^{\hbar/(k_BT)}d\tau\int_0^{\hbar/(k_BT)}d\tau'\\ &\quad \times G_T(\tau-\tau')\dot\varphi_i\dot\varphi_j, \end{aligned} \qquad (3)$$

where the kernel $G_T(\tau)$ is determined as

$$\begin{aligned} G_T(\tau) &= \frac{k_BT}{\hbar}\sum_n\frac{e^{i\omega_n\tau}}{\omega_n^2+\omega_R^2},\\ \omega_n &= n(\pi k_BT)/\hbar, \quad n = 0,\pm1,\pm2\ldots; \end{aligned} \qquad (4)$$

and $\xi_i = \eta_i\sqrt{E_J/m}$ are dimensionless coupling constants.

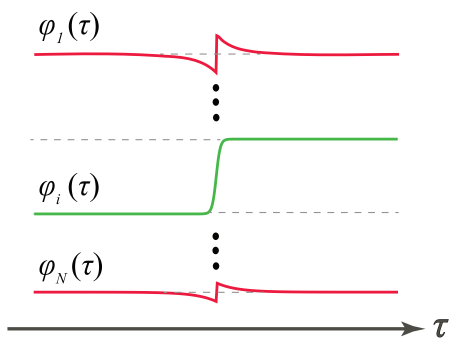

FIG. 2. (Color online) A typical saddle-point solution: $\{\varphi_k(\tau)\}$. It consists of a single instanton "kink" on ith qubit [solid green (thick) line] and "tails" on other qubits [solid red (thin) lines; the tails of 1 and Nth qubits are shown]. The kink describes the switching between two minima of a qubit potential while the tails caused by interaction describe small oscillations of $\varphi_k(\tau)$ around equilibrium positions.

The partition function is determined by saddle-point solutions that satisfy the equation

$$\frac{\ddot{\varphi}_i}{\omega_p^2} + \xi_i \sum_j \xi_j \int_0^{\beta\hbar} G_T(\tau-\tau')\ddot{\varphi}_j(\tau')d\tau' + \alpha_i\varphi_i - \frac{\varphi_i^3}{6} = 0. \tag{5}$$

Let us consider solutions of the following form: an instanton (anti-instanton) solution $f(\tau)$ on the lth qubit and perturbative "tails" on other qubits. This type of saddle-point solution is shown in Fig. 2. For the tails we linearize equations near the minimums of the potential $U(\varphi_i)$. In the absence of interaction the instanton (anti-instanton) solution is written as

$$f(\tau) = f_0(\tau-\tau_c) = \pm\sqrt{6\alpha_l}\tanh\left[\sqrt{\frac{\alpha_l}{2}}\omega_p(\tau-\tau_c)\right], \tag{6}$$

where τ_c is the instanton "center" time. Using Eqs. (5) and (6) we obtain the effective action of this solution as [see details in the Appendix]

$$S_{\rm eff}^l = S_0^l + \frac{1}{2}\xi_l^2 \int_0^{\beta\hbar} d\tau \int_0^{\beta\hbar} d\tau' G_1(\tau-\tau')\dot{f}_0(\tau)\dot{f}_0(\tau'), \tag{7}$$

where S_0^l is the action calculated for instanton solution $f_0(\tau)$ in the absence of interaction between qubits, and the Fourier transform of the kernel $G_1(\omega_n) = G_T(\omega_n)/[1+\kappa(\omega_n)]$, where $\kappa(\omega_n) = \sum_{j\neq l} \frac{\xi_j^2\omega^2 G_T(\omega_n)}{\omega_n^2/\omega_p^2 + 2\alpha_j}$.

Now considering multi-instanton solutions with the help of *noninteracting instanton (anti-instanton)* approximation [34], which is valid for rather weak interaction strength between qubits, one can calculate the partition function Z in a similar fashion to [35]. It is then written as $Z = \prod_{i=1}^N Z_i$, $Z_i = 2\cosh[\Delta_i/(k_BT)]$ and $\Delta_i \simeq \frac{\hbar}{\omega_p\sqrt{\alpha_i}}\exp(-S_{\rm eff}^i/\hbar)$. Calculating integrals over τ in Eqs. (3) and (7) we obtain

$$S_{\rm eff}^i = E_J 4\sqrt{2}\frac{\alpha_i^{3/2}}{\omega_p} + S_{\rm int}^i. \tag{8}$$

The coupling between qubits results in an enhancement of effective action $S_{\rm eff}^i$, and therefore, a decrease of average level splitting Δ_i. Moreover, the dispersion of qubit level splittings will be enhanced. The explicit value of $S_{\rm int}$ is determined by the parameter

$$\beta \simeq \left[2 + (N-1)\left\langle \xi_j^2/\alpha_j \right\rangle\right] \tag{9}$$

and the ratio of two frequencies: $\sqrt{\alpha_i}\omega_p$, i.e., the frequency of small oscillations on the bottom of the potential well, and ω_R. Here, the $\langle\cdots\rangle$ determines the averaging over a spread of qubit parameters ξ_i, α_i, and Δ_i. Explicit calculating integrals in (7) allows one to obtain

$$S_{\rm int}^i = \xi_i^2\alpha_i \frac{2\sqrt{2}E_J}{\omega_R}\begin{cases} \frac{3}{\sqrt{\beta}} & \text{if} \quad \beta\left(\frac{\sqrt{\alpha_i}\omega_p}{\omega_R}\right)^2 \gg 1 \\ \frac{\sqrt{\alpha_i}\omega_p}{\omega_R} & \text{if} \quad \beta\left(\frac{\sqrt{\alpha_i}\omega_p}{\omega_R}\right)^2 \ll 1. \end{cases} \tag{10}$$

III. TIME-DEPENDENT QUANTUM-MECHANICAL CORRELATION FUNCTION

In order to analyze the quantum dynamics of an array of interacting qubits we obtain the time-dependent correlation function of a single qubit, i.e., $C_i(t) = \langle\varphi_i(t)\varphi_i(0)\rangle$. In the noninteracting instanton (anti-instanton) approximation we can write $\varphi_i(\tau)$ as a sum:

$$\varphi_i(\tau) = f_i(\tau) + \sum_{j\neq i}\widetilde{\varphi}_i^j(\tau), \tag{11}$$

where $f_i(\tau) = \sum_k f_0(\tau - \tau_i^{(k)})$ consists of alternating instanton and anti-instanton "kinks," $\tau_i^{(k)}$ are randomly distributed instanton center times, and $\widetilde{\varphi}_i^j(\tau)$ correspond to tails from instantons (anti-instantons) on the jth qubit. The typical solution $\{\varphi_i(\tau)\}$ for a single instanton and many instantons (anti-instantons) are shown in Figs. 2 and 3.

The correlation function $C_i(\tau)$ is written as

$$C_i(\tau) = \langle f_i(\tau)f_i(0)\rangle + \sum_{j\neq i}\left\langle\widetilde{\varphi}_i^j(\tau)\widetilde{\varphi}_i^j(0)\right\rangle. \tag{12}$$

Following Ref. [35] the first term in the right-hand part of Eq. (12) is obtained as [the details of calculation are presented in the Appendix]

$$C_i^0(\tau) = \varphi_0^2\frac{\cosh\left(\left[\frac{\hbar}{k_BT} - 2\tau\right]\Delta_i\right)}{\cosh[\Delta_i/(k_BT)]}, \tag{13}$$

where $\pm\varphi_0 = \pm\sqrt{6\alpha_i}$ are the minima of the double-well potential $U(\varphi_i)$. Carrying out the analytical continuation to the real time we obtain in the low-temperature limit, i.e.,

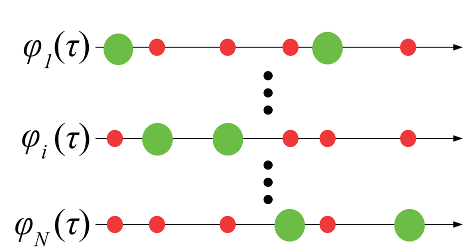

FIG. 3. (Color online) A typical saddle-point solution $\varphi_k(\tau)$ consisting of a large amount of randomly distributed instantons (anti-instantons) $f_i(\tau)$ [large (green) circles] and corresponding tails [small (red) circles] $\varphi_i^j(\tau)$ are shown.

$k_B T \ll \Delta_i$, the correlation function of noninteracting qubits as

$$C_i^0(t) = \varphi_0^2 e^{-2i\Delta_i t/\hbar}. \tag{14}$$

This result indicates the presence of quantum beating oscillations with N different frequencies, $\omega_i = \Delta_i/\hbar$, in the system.

However, there is another contribution to the correlation function of the ith qubit stemming from the tails of instantons (anti-instantons) occurring on other qubits. Such a contribution shown in Figs. 2 (a single instanton solution) and 3 [many instanton (anti-instanton) solution], is written as [it is a solution of Eq. (5) linearized around the instantons (anti-instantons)]

$$\tilde{\varphi}_i^j(\tau) = \frac{k_B T}{2\pi\hbar}\sum_n \int d\tau_1 G(\omega_n) e^{i\omega_n(\tau-\tau_1)} f_j(\tau_1), \tag{15}$$

where

$$G(\omega) = -\frac{\xi_i\xi_j\omega^2}{(2\alpha_i + (\omega/\omega_p)^2)\left[\omega^2+\omega_R^2+(N-1)\langle\xi_j^2/(2\alpha_j)\rangle\omega^2\right]}. \tag{16}$$

Substituting (15) in (12) and taking into account that $C_j^0(\tau) = \langle f_j(\tau)f_j(0)\rangle$ we obtain

$$\langle\tilde{\varphi}_i^j(\tau)\tilde{\varphi}_i^j(0)\rangle = \varphi_0^2\sum_n G(\omega_n)G(-\omega_n) \times \int d\tau_1 e^{i\omega_n(\tau-\tau_1)} C_j^0(\tau_1). \tag{17}$$

The quantum-mechanical dynamics is determined by the renormalized frequency of the resonator $\tilde{\omega}_R = \omega_R\sqrt{2/\beta}$. This renormalization occurs due to qubits introducing additional inductance to the resonator because of the coupling. It can be described as an effective homogeneous change of the resonator inductance per unit length $L_0 \to L_0\beta/2$. In the limit of $\frac{\sqrt{\alpha_i}\omega_p}{\tilde{\omega}_R} \gg 1$ the kernel $G(\omega)$ is simplified as $G(\omega) = -\frac{\xi_i\xi_l}{\beta\alpha_i}\frac{\omega^2}{[\omega^2+\tilde{\omega}_R^2]}$.

Carrying out the analytical continuation to the real time [36] we obtain the time-dependent correlation function in the following form:

$$C_i(t) = C_i^0(t) + C_1^{\rm col}(t) + C_2^{\rm col}(t),$$

where the time-dependent correlation functions $C_{1,2}^{\rm col}(t)$ are expressed as

$$C_1^{\rm col}(t) = (N-1)\varphi_0^2\left(\frac{\xi_i^2}{\alpha_i\beta}\right)^2 \times \left\langle \frac{16\xi_l^2(\Delta_l/\hbar)^4}{\left(\tilde{\omega}_R^2 - 4(\Delta_l/\hbar)^2\right)^2 + 4\gamma^2(\Delta_l/\hbar^2)} e^{-(2i\Delta_l/\hbar)(t)}\right\rangle, \tag{18}$$

$$C_2^{\rm col}(t) = (N-1)\varphi_0^2\left(\frac{\xi_i^2}{\alpha_i\beta}\right)^2 e^{-[2i\tilde{\omega}_R+\gamma/2]t} \times \left\langle \frac{-i16\xi_l^2(\Delta_l/\hbar)\tilde{\omega}_R^2/\gamma}{4(\Delta_l/\hbar)^2 - \tilde{\omega}_R^2 + i\gamma\tilde{\omega}_R/2}\right\rangle, \tag{19}$$

where γ is a phenomenological parameter describing dissipation in the resonator. This parameter allows one to keep finite the resonant term in the correlation functions $C_{1,2}^{\rm col}(t)$ as $\tilde{\omega}_R \simeq \Delta_l$. The correlation functions $C_{1,2}^{\rm col}(t)$ determine two *collective quantum-mechanical oscillations* with two frequencies, namely, the energy level splitting averaged over an ensemble of qubits, $\omega_1 = \bar{\Delta}/\hbar$ and self-frequency of the resonator renormalized by interaction $\omega_2 = \tilde{\omega}_R$. The physical origin of these collective oscillations is the excitation of quantum oscillations of individual qubits by coherent quantum beatings of other qubits in a system of globally coupled qubits. Moreover, oscillations with frequency ω_1 decay in time due to the dissipation of qubits and to a spread of qubit parameters. The second type of oscillations with the frequency ω_2 decays in time due to the dissipation of resonator γ, only. The amplitudes of these oscillations enhance strongly in the resonant case as $\tilde{\omega}_R \simeq \Delta_l$. Such an enhancement can also lead to a suppression of the double-well potential barrier for the Josephson phase, and therefore, to an increase of a particular level splitting Δ_l. This effect is similar to a well-known microwave induced enhancement of macroscopic quantum tunneling in Josephson junctions [39]. Note that in the case $N \to \infty$ corrections to $C(t) = \sum_i C_i(t)$ due to interaction stay finite because β in the denominators of (18) and (19) is proportional to N in the large N limit [see (9)].

IV. TRANSMISSION LINE EXPERIMENT

The correlation function $C_i(t)$, and therefore, the characteristic frequencies ω_i, ω_1, and ω_2 can be directly obtained in a so-called "dispersive readout." In such a setup a frequency dependent transmission coefficient $D(\omega)$ of a transmission line (TL) inductively coupled to a low-dissipative resonator coupled to a set of qubits is measured (see Fig. 4).

The electromagnetic waves in TL are characterized by coordinate and time dependent charge distribution $q(y,t)$ satisfying the wave inhomogeneous equation:

$$\frac{1}{c^2}\frac{\partial^2 q(y,t)}{\partial t^2} - \frac{\partial^2 q(y,t)}{\partial y^2} = \kappa\delta(y-y_0)\dot{Q}(t), \tag{20}$$

where $Q(t)$ is the charge distribution of a single mode in the resonator. The $Q(t)$ satisfies to an inhomogeneous dynamic

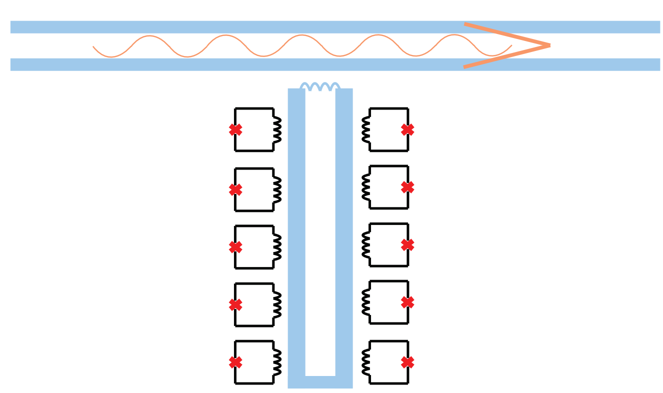

FIG. 4. (Color online) A schematic of the experimental setup: A low-dissipative TL is inductively coupled to the investigated system, i.e., the array of qubits incorporated in a resonator.

equation

$$\ddot{Q}+\omega_R^2 Q=\tilde{\kappa}\dot{q}(y_0,t)-\sum_i \xi_i\langle\dot{\varphi}_i\rangle. \tag{21}$$

Here κ and $\tilde{\kappa}$ are the corresponding coupling parameters, and $\langle\cdots\rangle$ is the quantum-mechanical average. On the other hand $\langle\dot{\varphi}_i\rangle$ is the linear response of rf SQUIDs to an external perturbation caused by waves in the transmission line. Now let us assume a wave with frequency ω is passing through the line, i.e., $q(y,t)=q(y)e^{i\omega t}$ and $Q(t)=Q_0 e^{i\omega t}$:

$$-\frac{\partial^2 q(y)}{\partial y^2}-\kappa\delta(y-y_0)i\omega Q_0=\frac{\omega^2}{c^2}q,$$
$$-\omega^2 Q_0+\omega_R^2 Q_0=\tilde{\kappa}i\omega q(y_0)+i\sum_i \omega\xi_i\langle\varphi_i\rangle_\omega.$$

According to Kubo's formula [40] the quantum-mechanical linear response of qubits to an external perturbation of the form $[i\xi_i\omega Q_0 e^{i\omega t}]\hat{\varphi}_i$ can be written as

$$\langle\varphi_i\rangle_\omega=i\omega\xi_i Q_0\tilde{C}_i(\omega),$$

where $\tilde{C}_i(\omega)=\int dt\ e^{i\omega t}\,\mathrm{Im}\,C_i(t)$ is the Fourier transformation of the *response function*.

Inserting the result into the resonator equation of motion one obtains Q_0:

$$Q_0=i\tilde{\kappa}\omega q(y_0)/F(\omega),$$

where $F(\omega)$ is

$$F(\omega)=-\omega^2+\omega_R^2-\chi\omega^2\tilde{C}(\omega)$$

and $\tilde{C}(\omega)=\sum_i \tilde{C}_i(\omega)$, with $\chi=\xi_i^2$. Now we turn to the transmission line equation, substituting the value of Q_0:

$$-\frac{\partial^2 q(y)}{\partial y^2}-\kappa\tilde{\kappa}\omega^2\delta(y-y_0)q(y_0)/F(\omega)=\frac{\omega^2}{c^2}q(y).$$

For the transmission line we have a Schrödinger equation with the δ-function potential. The solution of this problem is well known: $q(y)=e^{i\omega y/c}+Re^{-i\omega y/c}$ for $y<y_0$ and $Te^{i\omega y/c}$ for $y>y_0$, where reflection and transmission amplitudes are

$$T=\frac{2i\omega/c}{\kappa\tilde{\kappa}\omega^2/F(\omega)+2i\omega/c},$$
$$R=\frac{-\kappa\tilde{\kappa}\omega^2/F(\omega)}{\kappa\tilde{\kappa}\omega^2/F(\omega)+2i\omega/c}.$$

One can now determine the intensity and phase shift of the transmitted signal:

$$D(\omega)=|T|^2=\frac{4(\omega/c)^2}{[\kappa\tilde{\kappa}\omega^2/F(\omega)]^2+4(\omega/c)^2},$$
$$\Phi_D=\arctan\frac{\kappa\tilde{\kappa}\omega^2}{2F(\omega)\omega/c}.$$

Now one can see that for the frequencies ω_{res} satisfying the condition [which means $F(\omega)=0$]

$$\left(\omega_{\mathrm{res}}^2-\omega_R^2\right)=\chi\tilde{C}(\omega_{\mathrm{res}}), \tag{22}$$

the transmission $D(\omega)$ shows the resonant drop.

The Fourier transformation of the response function $C(\omega)$ has the resonant form [see Eqs. (14), (18), and (19)], and the resonant drops in the dependence of $D(\omega)$ are fingerprints of quantum-mechanical oscillations, i.e., quantum beatings of individual qubits (in the absence of interactions between qubits) and the collective oscillations.

Next, we show that in the absence of interaction between qubits and in the case as all qubits have the same parameters, i.e., $\Delta_i=\Delta$, our model reproduces the well-known Tavis-Cummings result [28] for avoided crossing level splittings. Indeed, the response function in this case displays a single resonance:

$$\tilde{C}(\omega)=\frac{2N\varphi_0^2\omega_q}{\omega_q^2-\omega^2}, \tag{23}$$

where $\omega_q=2\Delta/\hbar$. Substituting (23) in (22) one obtains the splitting of resonant frequencies as

$$-\omega_{\mathrm{res}}^2+\omega_R^2-\frac{2\chi N\varphi_0^2\omega_q}{\omega_q^2-\omega_{\mathrm{res}}^2}=0,\quad \omega_{\mathrm{res}}^2=\frac{1}{2}\left[\omega_q^2+\omega_R^2+2\chi N\varphi_0^2\omega_q\pm\sqrt{\left(\omega_q^2-\omega_R^2\right)^2+4\left(\omega_q^2+\omega_R^2\right)\chi\varphi_0^2\omega_q N+4\varphi_0^4\chi^2\omega_q^2N^2}\right],$$

Thus, in the case of identical qubits the splitting between two resonant drops is strongly enhanced ($\propto\sqrt{N}$). Moreover, in a realistic case of nonidentical qubits an enhanced splitting can be considered as a fingerprint of collective (synchronized) quantum beating [31].

V. CONCLUSIONS

In conclusion, we have shown that an array of strongly coupled qubits can display coherent collective quantum oscillations. We have considered a particular example of an array of superconducting qubits (rf SQUIDs) incorporated into a resonator. In such system a long-range interaction (a global coupling) can be provided by emission (absorption) of virtual photons in the resonator. By analyzing quantum-mechanical correlation functions we have obtained that beyond quantum beating oscillations with different frequencies, $\omega_i=\Delta_i/\hbar$, there are two collective quantum-mechanical oscillations with two frequencies, ω_1 and ω_2. These collective oscillations appear in the presence of a long-range coupling between qubits, and they are induced by coherent quantum beatings occurring in a whole system. The characteristic frequencies of these oscillations can be directly measured through, e.g., the resonant drops of transmission coefficient $D(\omega)$ of electromagnetic field propagating in the transmission line coupled to a system [see Eq. (22)]. The observation of these collective quantum-mechanical modes will provide evidence of synchronized quantum dynamics in a system of strongly interacting qubits.

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APPENDIX

1. A single instanton saddle-point solution and an effective action of *N* qubits with a long-range interaction

Classical equations of motion for the action S_{eff} are given by

$$\frac{\ddot{\varphi}_i}{\omega_p^2} + \xi_i \sum_j \xi_j \int_0^{\beta\hbar} G_T(\tau - \tau')\ddot{\varphi}_j(\tau')d\tau' + \alpha_i \varphi_i - \frac{\varphi_i^3}{6} = 0. \tag{A1}$$

We seek a saddle-point solution consisting of an instanton solution $f(\tau)$ on the lth rf SQUID and a perturbative tail $\varphi_i(\tau)$ on the others near the extremum positions of the potential $\pm\sqrt{6\alpha_i}$. For the tail we linearize the equations $\alpha_i\varphi_i - \frac{\varphi_i^3}{6} \approx -2\alpha_i\varphi_i$. We also make use of the Fourier transformation:

$$\varphi_i(\tau) = \frac{1}{\beta\hbar}\sum_n e^{i\omega_n\tau}\varphi_i(\omega_n), \qquad i \neq l.$$

For $i \neq l$ we obtain (we write just ω instead of ω_n)

$$-\frac{\omega^2}{\omega_p^2}\varphi_i(\omega) - \xi_i \sum_j \xi_j \omega^2 \varphi_j(\omega) g_T(\omega) - 2\alpha_i\varphi_i(\omega) = 0. \tag{A2}$$

Multiplying by ξ_i and summing over $i \neq l$ one arrives at

$$\sum_{i\neq l}\xi_i\varphi_i(\omega) = \frac{-\kappa\xi_l f(\omega)}{1+\kappa}, \quad \kappa = \sum_{j\neq l}\frac{\xi_j^2\omega^2 g_T(\omega)}{\omega^2/\omega_p^2 + 2\alpha_j}. \tag{A3}$$

Applying (A3) to (5) we have the following self-consistent equation for $f\tau$:

$$\frac{\ddot{f}}{\omega_p^2} + \xi_l^2 \int_0^{\beta\hbar} G_1(\tau-\tau')\ddot{f}(\tau')d\tau' + \alpha_l f - \frac{f^3}{6} = 0, \tag{A4}$$

where G_1 is defined by its Fourier transform: $g_1(\omega) = g_T(\omega)/(1+\kappa)$. Now let us return to the effective action S_{eff}:

$$\begin{aligned} S_{\text{eff}}^E &= S_l + S_{\text{int}} + E_J\left\{\sum_{i\neq l}\int_0^{\beta\hbar} d\tau\left[\frac{(\dot{\varphi}_i)^2}{2\omega_p^2} - \frac{3\alpha_i}{2} + \alpha_i\varphi_i^2\right]\right\} = S_l + S_{\text{int}} + E_J\left\{\sum_{i\neq l}\int_0^{\beta\hbar} d\tau\left[-\frac{\varphi_i\ddot{\varphi}_i}{2\omega_p^2} + \alpha_i\varphi_i^2\right]\right\} \\ &= S_l + S_{\text{int}} + E_J\sum_{i\neq l}\int_0^{\beta\hbar} d\tau\, \xi_i\frac{\varphi_i(\tau)}{2}\sum_j \xi_j\int_0^{\beta\hbar} G_T(\tau-\tau')\ddot{\varphi}_j(\tau')d\tau' \\ &= E_J\left\{\int_0^{\beta\hbar} d\tau\left[\frac{(\dot{f})^2}{2\omega_p^2} - \frac{\alpha_l f^2}{2} + \frac{f^4}{4!}\right] + \frac{1}{2}\xi_l^2\int_0^{\beta\hbar} d\tau\int_0^{\beta\hbar} d\tau' G_1(\tau-\tau')\dot{f}(\tau)\dot{f}(\tau')\right\}. \end{aligned} \tag{A5}$$

Considering $f = f_0 + \widetilde{f}$, where f_0 is the instanton solution in the absence of the effective interaction term and $\widetilde{f}$ is small on the order of ξ_l^2, we arrive at Eq. (7):

$$S_{\text{eff}}^E = S_{\text{inst}}^0 + \frac{1}{2}\xi_l^2\int_0^{\beta\hbar} d\tau\int_0^{\beta\hbar} d\tau' G_1(\tau-\tau')\dot{f}_0(\tau)\dot{f}_0(\tau').$$

2. The time-dependent correlation function $C_i^0(t)$ of a single qubit

The time-dependent correlation function of a single qubit (without any interactions) $C_i^0(t) = \langle f_i(t)f_i(0)\rangle$ can be obtained as follows: we write $f_i(\tau) = \sum_k f_0(\tau - \tau_i^{(k)})$ and assume the *noninteracting instanton approximation*. It is valid as $\Delta_i \gg k_B T$. One can also see that for configurations with k instantons and k anti-instantons in the interval $[0,\tau]$, $f_i(\tau)f_i(0) = \varphi_0^2$ and for those with k instantons and $k-1$ anti-instantons, $f_i(\tau)f_i(0) = -\varphi_0^2$.

Following [35] we can replace the path integral with the sum

$$\begin{aligned} \langle \hat{f}_i(\tau)\hat{f}_i(0)\rangle &\approx \frac{2}{Z}\sum_{n=0}^{\infty}(\Delta)^{2n}\int_0^{\beta\hbar} dt_{2n}\int_0^{t_{2n}} dt_{2n-1}\cdots\int_0^{t_2} dt_1 \\ &\times \prod_{i=1}^{2n}\varphi_0^2 \operatorname{sgn}(t_i - \tau) = \frac{2}{Z}\sum_{n=0}^{\infty}(\Delta)^{2n}\varphi_0^2 I_{2n}(\beta\hbar), \end{aligned} \tag{A6}$$

where $\Delta = (\hbar/\tau_0)\exp(-S_{\text{inst}}/\hbar)$, τ_0 is the instanton rise time, t_{2k+1} and t_{2k} are instanton and anti-instanton center times, respectively, and the product of sign functions gives the right sign to every path contribution according to the rule described above. Now we proceed to calculate I_{2n}:

$$I_{2n}(\beta\hbar) = \int_0^{\beta\hbar} dt_{2n}\int_0^{t_{2n}} dt_{2n-1}\cdots\int_0^{t_2} dt_1\prod_{i=1}^{2n}\operatorname{sgn}(t_i - \tau). \tag{A7}$$

One can see that integrand is invariant under permutations of $t_1 \cdots t_{2n}$. Then we can write $I_{2n}(\beta\hbar)$ in a symmetric form:

$$I_{2n}(\beta\hbar) = \frac{1}{(2n)!} \sum_{\sigma} \int_0^{\beta\hbar} d[\sigma(t_{2n})] \int_0^{\sigma(t_{2n})} d[\sigma(t_{2n-1})] \cdots \int_0^{\sigma(t_2)} d[\sigma(t_1)] \prod_{i=1}^{2n} \mathrm{sgn}(t_i - \tau), \tag{A8}$$

where σ is an element of the permutation group and the sum is calculated over all possible permutations. One can now see that each term in the sum is an integral over one part of a $2n$-dimensional cube, and together the regions of integration simply add up to become the whole $2n$-dimensional cube. This leads us to

$$I_{2n}(\beta\hbar) = \frac{1}{(2n)!} \int_0^{\beta\hbar} dt_{2n} \int_0^{\beta\hbar} dt \cdots \int_0^{\beta\hbar} dt_1 \prod_{i=1}^{2n} \mathrm{sgn}(t_i - \tau) = \frac{1}{(2n)!} \left(\int_0^{\beta\hbar} dt\, \mathrm{sgn}(t - \tau) \right)^{2n} = \frac{(\beta\hbar - 2\tau)^{2n}}{(2n)!}. \tag{A9}$$

Now we can evaluate the correlation function using $Z = 2\cosh(\beta\hbar\Delta)$:

$$C_i(\tau) \approx \frac{2}{Z} \sum_{n=0}^{\infty} (\Delta)^{2n} \varphi_0^2 \frac{(\beta\hbar - 2\tau)^{2n}}{(2n)!} = \varphi_0^2 \frac{\cosh([\beta\hbar - 2\tau]\Delta)}{\cosh(\beta\hbar\Delta)}. \tag{A10}$$

The analytic continuation to the real time t allows one to obtain

$$C_i^0(t) = \varphi_0^2 \frac{\cosh([\beta\hbar - 2i(t)]\Delta)}{\cosh(\beta\hbar\Delta)}. \tag{A11}$$

In the limit of $\beta\hbar \to \infty$ one obtains [see Eq. (14)]

$$C_i^0(t) \approx \varphi_0^2 e^{-2i\Delta t/\hbar}. \tag{A12}$$

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