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Generation of non-classical photon states in superconducting quantum metamaterials

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Abstract

We report a theoretical study of diverse non-classical photon states that can be realized in superconducting quantum metamaterials. As a particular example of superconducting quantum metamaterials, an array of SQUIDs incorporated in a low-dissipative transmission line (resonant cavity) will be studied. This system will be modeled as a set of two-level systems (qubits) strongly interacting with resonant cavity photons. We predict and analyze a second-order phase transition between incoherent (the high-temperature phase) and coherent (the low-temperatures phase) states of photons. In the equilibrium state the partition function Z of the electromagnetic field (EF) in the cavity is determined by the effective action $S_{\mathrm{eff}}\{P(\tau)\}$ that, in turn, depends on the imaginary time dependent momentum of the photon field $P(\tau)$. We show that the order parameter of this phase transition is the $P_0(\tau)$ minimizing the effective action of the whole system. In the incoherent state, the order parameter $P_0(\tau) = 0$ but at low temperatures we obtain various coherent states characterized by non-zero values of $P_0(\tau)$. This phase transition in many aspects resembles the Peierls metal–insulator and the metal–superconductor phase transitions. The critical temperature of such a phase transition T^* is determined by the energy splitting of two-level systems Δ, the number of SQUIDs in the array N and the strength of the interaction η between SQUIDs and photons in the cavity.

(Some figures may appear in colour only in the online journal)

1. Introduction

Great attention has been devoted to the theoretical and experimental study of novel quantum metamaterials [1–3]. These systems consist of a large number of solid-state elements (qubits), i.e. two-level systems showing diverse coherent quantum phenomena, e.g. quantum beating (oscillations) between two distinguished states and, in the presence of externally applied radiation, microwave induced Rabi oscillations, Ramsey fringes etc [4]. To obtain such coherent quantum-mechanical behavior in single qubits, the dissipation and decoherence have to be small enough [4]. Moreover, in order to observe novel collective coherent quantum effects in metamaterials a strong long-range interaction between single elements has to be provided by the surrounding media.

Various superconducting systems, e.g. arrays of Josephson junctions, RF SQUIDs, multiple-junction superconducting quantum interferometers, just to name a few, incorporated in a low-dissipative (superconducting) transmission line are extremely suitable in order to realize such quantum metamaterials.

Indeed, diverse SQUIDs subject to an externally applied magnetic field can be modeled by a double-well potential for a single degree of freedom, i.e. the Josephson phase [4]. Although the quantum-mechanical versus classical description of such systems is still under debate [5, 6], we believe that at low temperatures, as the dissipative effects are small and as the potential barrier between two classical states is rather small, the coherent tunneling between two states results in two low-lying energy levels. These energy levels are well separated from other levels and the energy level difference Δ is tunable by an externally applied magnetic field. Such energy level diagrams and the coherent quantum beating between two states have been observed experimentally in diverse superconducting systems [7–10] and, therefore, superconducting systems show a good potential to be implemented as qubits.

A strong long-range interaction between well-separated qubits is provided by a transmission line through emission (absorption) of virtual photons. This type of interaction was proposed in [11–15] and realized in experiments with single qubits incorporated in a resonator [16, 17]. It has been shown that a strong interaction between well-separated qubits results in an enhancement of quantum-mechanical tunneling [15, 18, 19] and suppression of decoherence induced by a spread of parameters of qubits [20]. The measurements of the frequency dependent transmission (reflection) coefficient of electromagnetic field (EF) propagating through the transmission line provide a convenient method to observe coherent quantum phenomena in such metamaterials [21, 22].

On the other hand, a strong interaction of qubits with EF can result in different states of photons in the cavity. Indeed, in the absence of interaction with qubits the unique photon state of the cavity is an incoherent, chaotic state of photons characterized by a well-known Planck distribution, i.e. $\langle\hat{E}\rangle = 0$, $\langle(\hat{E})^2\rangle \propto [\mathrm{e}^{\frac{\hbar\omega_0}{k_B T}} - 1]^{-1}$, where E is the electric field of radiation, $\langle\cdots\rangle$ is the quantum-mechanical average, and ω_0 is the frequency of the cavity mode. A strong interaction of EF with qubits leads to the effective enhancement of the difference in energy levels of the qubits which, in turn, changes EF in the cavity. Thus, one can expect that in a resonant cavity strongly interacting with an array of qubits different states of photons can be observed. In this paper we analyze possible quantum-mechanical states of photons emerging in the resonant cavity strongly interacting with an array of qubits. Notice here that a similar analysis of the photon states of the cavity interacting with an unbiased array of Josephson junctions has been reported in [23] in order to explain the strong radiation from a 2D-array of Josephson junctions observed in [24].

The paper is organized as follows: in section 2 we present a model of an array of SQUIDs incorporated in a low-dissipative transmission line and elaborate on the classical description of this system, i.e. the dynamic equations of motion and the Lagrange function; in section 3 a complete quantum-mechanical description of this model is provided. In section 4 using the great similarity with well-known phase transitions, e.g. the metal–ferromagnet [25], metal–superconductor [26] and the Peierls metal–insulator [27] transitions, we predict and analyze a second-order phase transition between the incoherent, chaotic state (the high-temperature phase) of photons and diverse coherent non-classical photon states (the low-temperature phase). Section 5 provides a discussion and conclusion.

2. Model and classical description of superconducting metamaterials

As a particular example of quantum metamaterials we consider here a system of RF SQUIDs incorporated in a low-dissipative transmission line. The RF SQUIDs will be modeled as tunable two-level systems. The inductive coupling between RF SQUIDs and the transmission line provides an interaction between cavity photons and two-level systems. Each RF SQUID is characterized by a Josephson phase $\varphi_i = 2\pi\Phi_i/\phi_0$, where Φ_i is the total flux in the superconducting loop of a SQUID and ϕ_0 is the flux quantum. An application of dc-magnetic field characterized by Φ_{ext} allows one to tune the potential relief of a Josephson phase φ_i from a single well up to a double-well potential. The set of RF SQUIDs is incorporated in a linear transmission line. The transmission line is characterized by two parameters L_0 and C_0, the inductance and capacitance per unit length, respectively. We also introduce the voltage $V(x)$ and current $I(x)$ distributions, where x is the coordinate along a transmission line. The inductive coupling, $M = \eta L_0$, provides an interaction between RF SQUIDs and the transmission line. The schematic of the system is presented in figure 1.

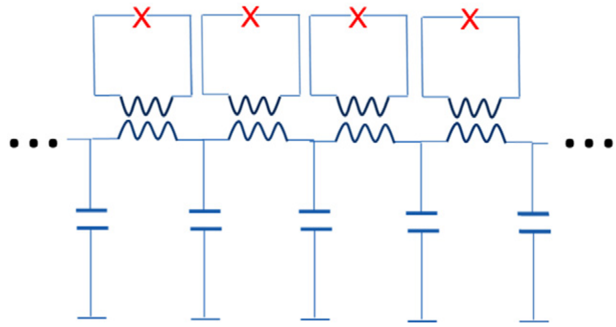

Figure 1. The schematic of an array of RF SQUIDs incorporated in a transmission line.

2.1. *Classical equations of motion: linear transmission line*

We start with the classical dynamic equations for a linear transmission line. It is

$$\frac{\partial V(x,t)}{\partial x} = L_0 \frac{\partial I(x,t)}{\partial t} \qquad (1)$$

and

$$\frac{\partial I(x,t)}{\partial x} = C_0 \frac{\partial V(x,t)}{\partial t}. \qquad (2)$$

These two equations are rewritten as

$$\frac{\partial^2 I(x,t)}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 I(x,t)}{\partial t^2}, \qquad (3)$$

where $L_0 C_0 = 1/c^2$. The electromagnetic standing waves can occur in this 1D cavity resonator. The wavevectors are determined by standard boundary conditions: $k_n = \pi n/\ell$, where ℓ is the size of a transmission line, $n = 1, 2, \ldots$. We will consider a transmission line with an extremely high quality factor, which was routinely obtained in superconducting transmission lines and, therefore, only one wavevector will be important in the dynamics of the coupled RF SQUIDs and EWs of the cavity.

2.2. Classical equations of motion: an individual SQUID

The classical dynamic equations for an individual SQUID are written as

$$I_{\mathrm{RF}}/I_{\mathrm{c}} = \sin(\varphi) + \frac{\alpha}{\omega_{\mathrm{p}}}\frac{\mathrm{d}\varphi}{\mathrm{d}t} + \frac{1}{\omega_{\mathrm{p}}^2}\frac{\mathrm{d}^2\varphi}{\mathrm{d}t^2}, \tag{4}$$

where I_{c} is the critical current of a Josephson junction, $1/\alpha^2$ is the McCumber parameter characterizing the dissipation of the SQUID and ω_{p} is the plasma frequency of a Josephson junction. On the other hand, the Josephson phase in a SQUID loop is satisfied by the following equation

$$\varphi = \varphi_{\mathrm{ext}} - \beta_L I_{\mathrm{RF}}/I_{\mathrm{c}}, \tag{5}$$

where β_L is the inductive (dimensionless) parameter of the SQUID, φ_{ext} corresponds to the sum of the externally applied dc-magnetic field and the ac-magnetic field induced by a current flowing along the transmission line. Thus, $\varphi_{\mathrm{ext}}^{\mathrm{dc}}$ allows one to tune the potential relief of the Josephson phase and $\varphi_{\mathrm{ext}}^{\mathrm{ac}} \propto I$ provides a coupling between the RF SQUID and a transmission line.

2.3. Classical equation of motion: coupled transmission line and SQUIDS

Inductive coupling between RF SQUIDs and the transmission line results in a particular change of classical equations of motion: first, equation (3) changes to

$$\frac{\partial^2 I(x,t)}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 I(x,t)}{\partial t^2} + \sum_i \frac{\eta}{c^2}\frac{\partial^2 I_{\mathrm{RF}}^{(i)}(t)}{\partial t^2}, \tag{6}$$

where η is the parameter characterizing a mutual inductance of RF SQUIDs and the transmission line; secondly, $\varphi_{\mathrm{ext}}^{\mathrm{ac}} = (\eta L_0/\phi_0) I(x_i, t)$.

2.4. Lagrangian of a superconducting metamaterial

The classical equations of motion can also be derived from the Lagrangian of a whole system

$$\begin{aligned} L = {} & \frac{L_0}{2}(\dot{Q}(x,t))^2 - \frac{1}{2C_0}\left(\frac{\partial Q(x,t)}{\partial x}\right)^2 + E_{\mathrm{J}}\sum_i \frac{1}{2\omega_{\mathrm{p}}^2}[\dot{\varphi}_i]^2 \\ & - \frac{1}{2\beta_L}[\varphi_i - \varphi_{\mathrm{ext}}^{\mathrm{dc}} + (\eta L_0/\phi_0)\dot{Q}(x,t)]^2 \\ & - (1-\cos\varphi_i), \end{aligned} \tag{7}$$

where $Q(x,t)$ is the charge variable characterizing a transmission line and $E_{\mathrm{J}} = \hbar I_{\mathrm{c}}/(2e)$ is the energy of the Josephson junction. To simplify this expression we consider the standing EWs with a single wavevector k_n only. In this case the charge variable $Q(x,t)$ has the form: $Q(x,t) = Q(t)\cos(k_n x)$. Substituting this expression in equation (7) we obtain

$$L = L_{\mathrm{ph}} + L_{\mathrm{SQUID}} + L_{\mathrm{int}},$$

where

$$L_{\mathrm{ph}} = m\left[\frac{1}{2}(\dot{Q}(t))^2 - \frac{c^2k_n^2}{2}(Q(t))^2\right], \qquad m = L_0\ell/2, \tag{8}$$

and

$$\begin{aligned} L_{\mathrm{SQUID}} = {} & E_{\mathrm{J}}\sum_i \frac{1}{2\omega_{\mathrm{p}}^2}[\dot{\varphi}_i]^2 - \frac{1}{2\beta_L}[\varphi_i - \varphi_{\mathrm{ext}}^{\mathrm{dc}}]^2 \\ & - (1-\cos\varphi_i), \end{aligned} \tag{9}$$

and

$$L_{\mathrm{int}} = -\sum_i (\eta L_0/\Phi_0)\frac{E_{\mathrm{J}}}{\beta_L}\varphi_i\cos(k_n x_i)\dot{Q}(t), \tag{10}$$

where x_i are the coordinates of RF SQUIDs along a transmission line.

3. Quantum description of superconducting quantum metamaterials

3.1. Photon Hamiltonian

Introducing the 'momentum' of a photon field $P(t)$ as $P = (L_0\ell/2)\dot{Q}(t)$ we obtain the Hamiltonian of a photon field in the following form:

$$H_{\mathrm{ph}} = \frac{P^2}{\ell L_0} + \frac{L_0\ell c^2 k_n^2}{4}[Q(t)]^2. \tag{11}$$

Next we introduce the operators of the boson field $\hat{b}$ and $\hat{b}^+$ as

$$\hat{Q}(t) = \sqrt{\frac{\hbar}{ck_nL_0\ell}}(\hat{b} + \hat{b}^+)$$

and

$$\hat{P}(t) = -\mathrm{i}\sqrt{\frac{\hbar ck_nL_0\ell}{4}}(\hat{b} - \hat{b}^+).$$

Using these new variables the photon Hamiltonian is written as

$$H_{\mathrm{ph}} = \hbar\omega_0(\hat{b}^+\hat{b} + 1/2), \qquad \omega_0 = ck_n. \tag{12}$$

3.2. RF SQUID Hamiltonian

We consider the macroscopic quantum dynamics of the RF SQUID when the potential energy has the form of double-well potential. In this case the Hamiltonian of a system of isolated RF SQUIDs is written as

$$H_{\mathrm{SQUID}} = \sum_i \tfrac{1}{2}[\Delta_i\hat{\sigma}_x + \epsilon_i\hat{\sigma}_z], \tag{13}$$

where σ_z and σ_x are standard Pauli matrices, Δ_i and ϵ_i are the matrix element of tunneling and the energy difference between two potential wells, respectively. Notice here that such a Hamiltonian can also be used for more complex qubits, e.g. phase qubits, flux qubits etc, where parameters Δ_i and ϵ_i are determined by the physical properties of the corresponding qubits. For qubits based on RF SQUIDs we obtain

$$\Delta \propto \hbar\omega_{\mathrm{p}}(1-1/\beta_L)^{1/2}\mathrm{e}^{-\frac{4\sqrt{2}E_{\mathrm{J}}}{\hbar\omega_{\mathrm{p}}}(1-1/\beta_L)^{3/2}} \tag{14}$$

and

$$\epsilon \propto \pi\sqrt{6}(1-1/\beta_L)^{1/2}(\phi_{\mathrm{ext}}/\phi_0 - 1). \tag{15}$$

Moreover, the parameters Δ_i and ϵ_i can fluctuate from one qubit to other one.

3.3. Interaction Hamiltonian

The equilibrium dynamics of a Josephson phase in imaginary time representation can be presented as rare jumps (the instanton type of solution) between two equilibrium positions [28]. Using this property we obtain the interaction Hamiltonian as follows:

$$H_{\rm int} = i\sum_i \xi_i \hat{\sigma}_z(\hat{b} - \hat{b}^+),$$
$$\xi_i = \frac{E_{\rm J}\eta(\delta\varphi)\sqrt{\hbar c k_n L_0 \ell}}{\beta_L \phi_0} \tag{16}$$

where $\delta\varphi$ is the phase difference between two equilibrium positions of a Josephson phase.

3.4. Effective action

In order to study the various photon states arising in superconducting quantum metamaterials we write the partition function Z in the form of a functional integral as

$$Z = \int DQD\{\varphi_i\}\exp(-S\{Q,\varphi_i\}), \tag{17}$$

where S is the action of EF interacting with an array of two-level systems. Integrating equation (17) over $\{\varphi_i\}$ we obtain the effective action $S_{\rm eff}$ in the following form [29]:

$$S_{\rm eff}[Q(\tau)] = \frac{1}{\hbar}\int_0^{\hbar/(k_{\rm B}T)} {\rm d}\tau \frac{m}{2}[\dot{Q}^2 + \omega_0^2 Q^2]$$
$$- k_{\rm B}T\sum_i \ln\left[\cosh\frac{\alpha_i\{Q\}}{2k_{\rm B}T}\right], \tag{18}$$

where $\alpha_i\{Q(\tau)\}$ are the positive Floquet eigenvalues of arrays of two-level systems in the presence of the potential $Q(\tau)$, which is periodic in imaginary time. Notice here that in the absence of interaction with the field $Q(\tau)$, i.e. as $\xi_i = 0$, the Floquet eigenvalues are $\alpha_i(0) = \sqrt{\Delta_i^2 + \epsilon_i^2}$ and the minimum of $S_{\rm eff}[Q(\tau)]$ occurs as $Q = 0$.

4. Phase transitions in states of photons

Since the interaction term in the Lagrange function $L_{\rm int}$ depends on the τ-derivative of Q, i.e. $\dot{Q}$, we introduce a new variable of the momentum of a photon field $P(\tau) = m\dot{Q}$ and rewrite the effective action $S_{\rm eff}$ as

$$S_{\rm eff}[P(\tau)] = \frac{1}{\hbar}\int_0^{\hbar/(k_{\rm B}T)} {\rm d}\tau \frac{1}{2m}\left[P^2 + \frac{1}{\omega_0^2}\dot{P}^2\right]$$
$$- k_{\rm B}T\sum_i \ln\left[\cosh\frac{\alpha_i\{P\}}{2k_{\rm B}T}\right]. \tag{19}$$

Moreover, the Floquet eigenvalues $\alpha_i(P)$ are determined from the following equation:

$$(\partial_\tau + \hat{H}_P^i)\psi_i = 0;$$
$$\psi_i(\tau + 1/k_{\rm B}T) = {\rm e}^{-\alpha_i/k_{\rm B}T}\psi_i(\tau) \tag{20}$$

and the Hamiltonian H_P^i is:

$$\hat{H}_P^i = \begin{pmatrix} \epsilon_i + \tilde{\eta}_i P(\tau) & \Delta_i \\ \Delta_i & -(\epsilon_i + \tilde{\eta}_i P(\tau)) \end{pmatrix}, \tag{21}$$

where the parameter determining the interaction between cavity modes and two-level systems $\tilde{\eta}_i = \frac{\eta(\delta\varphi_i)E_{\rm J}}{\Phi_0\beta_L}\cos(k_n x_i)$.

Next, we obtain the periodic in imaginary time representation function $P_0(\tau + 1/k_{\rm B}T) = P_0(\tau)$, minimizing the effective action, as a solution of the following equation:

$$\frac{P_0}{m} + \frac{1}{m\omega_0^2}\ddot{P}_0 = \sum_i \frac{\partial\alpha_i}{\partial P}\tanh\left(\frac{\alpha_i\{P_0\}}{2k_{\rm B}T}\right). \tag{22}$$

4.1. Classical second-order phase transition

First we consider a particular case as the momentum of a photon field P_0 does not depend on τ. The Floquet eigenvalues are determined as $\alpha_i(P) = \sqrt{(\epsilon_i + \tilde{\eta}_i P_0)^2 + \Delta_i^2}$ and the equation for P_0 reads as

$$\frac{P_0}{m} = \sum_i \tilde{\eta}_i \frac{(\epsilon_i + \tilde{\eta}_i P_0)}{\sqrt{(\epsilon_i + \tilde{\eta}_i P_0)^2 + \Delta_i^2}}$$
$$\times \tanh\left[\frac{\sqrt{(\epsilon_i + \tilde{\eta}_i P_0)^2 + \Delta_i^2}}{2k_{\rm B}T}\right]. \tag{23}$$

In the simplest case as $\epsilon_i = 0$ and $\tilde{\eta}_i = \tilde{\eta}$ we obtain the self-consistent equation

$$P_0\left[\frac{1}{m\tilde{\eta}^2} - \sum_i \frac{1}{\sqrt{\tilde{\eta}^2 P_0^2 + \Delta_i^2}}\tanh\left(\frac{\sqrt{\tilde{\eta}^2 P_0^2 + \Delta_i^2}}{2k_{\rm B}T}\right)\right]$$
$$= 0. \tag{24}$$

At high temperatures, $T > T^\star$, this equation has a single solution $P_0 = 0$. Such a high-temperature phase corresponds to the incoherent, chaotic state of a photon field. However, at low temperatures, $T < T^\star$, equation (24) has two non-zero solutions of $\pm P_0$ and the coherent state of a photon field can be realized. Therefore, we obtain a second-order type of phase transition in the states of photon field interacting with a set of two-level systems. The critical temperature $T^\star$ depends on the distribution of the energy differences of two-level system Δ_i. If such a distribution is a narrow one, i.e. $\Delta_i \simeq \Delta$, we obtain the value of $T^\star$ as

$$T_n^\star = \frac{m\tilde{\eta}^2 N}{k_{\rm B}}, \tag{25}$$

where N is the total number of two-level systems. Such a phase transition occurs only if $\Delta_0 < k_{\rm B}T_n^\star$. At low temperatures P_0 reaches the maximum value of $P_0 = \pm m\tilde{\eta}N$. As one can see the maximum value of P_0 is proportional to N and this dependence also indicates the presence of the coherent state. The dependence of $P_0(T)$ for this case is shown in figure 2 (red solid line). Notice here that this case resembles a well-known metal–ferromagnet phase transition [25].

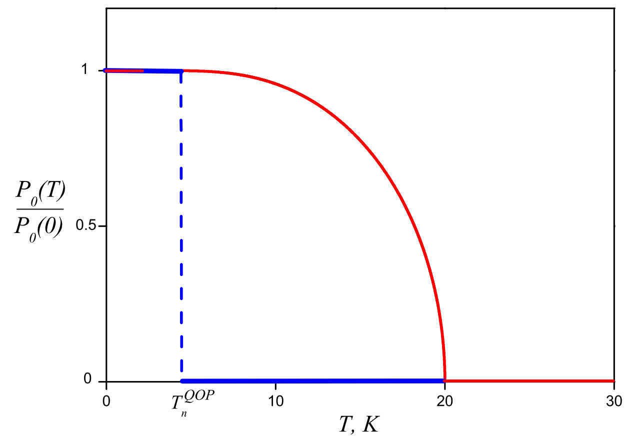

Figure 2. The classical and quantum phase transitions in the state of photons. The temperature dependences of the momentum of photon field $P_0(T)$ are shown: τ-independent P_0 (red line) and $P_0(\tau) \propto P_0(T)\sin(P_0(T)\tau)$ (blue line). The case of a narrow distribution of $\Delta_i \simeq \Delta_0$ is shown. The parameters $T_n^\star = 20$ K and $\Delta_0 = 4$ K were used.

In the opposite case of a wide distribution of Δ_i, e.g. from zero to upper cut-off Δ_0, the critical temperature is determined as

$$T_w^\star = \Delta_0 \mathrm{e}^{-\frac{\Delta_0}{m\tilde{\eta}^2 N}}. \tag{26}$$

This phase transition occurs if $\Delta_0 > m\tilde{\eta}^2 N$. At low temperatures P_0 reaches the maximum value of $|P_0| \simeq k_\mathrm{B} T_w^\star/\tilde{\eta}$. This phase transition resembles the superconductor–normal metal [26] and/or Peierls metal–insulator [27] transitions.

4.2. Quantum phase transitions

Here we consider a phase transition into a peculiar 'quantum ordered' state representing a quantum interference between the two semi-classical states $\pm P_0$ of the photon field, which are inversion symmetry related solutions of the self-consistency equation (24). Each of the $\pm P_0$ states separately describes a particular coherent state of the photon field in our model. The quantum ordered state was predicted in [29] for a system of electron–hole pairs coupled to a semi-classical spin-density wave fluctuation. In our present model it is described by the amplitude of the semi-classical photon field $P_0(\tau)$ or the 'quantum order parameter', which is the periodic in imaginary time solution of equation (22). Again we consider the simplest case as $\epsilon_i = 0$ and $\tilde{\eta}_i = \tilde{\eta}$ and apply the analytical solution found in [29, 30]:

$$\tilde{\eta} P_0(\tau) = 4nk_\mathrm{B}KTk_1 sn(4nk_\mathrm{B}KT\tau; k_1), \qquad K = K(k_1) \tag{27}$$

where $sn(\tau, k_1)$ is the Jacobi snoidal elliptic function, periodic in τ with period $1/(nk_\mathrm{B}T)$, $n = 1, 2, \ldots, K(k_1)$ is an elliptic integral of the first kind and positive integer n and real parameter $0 < k_1 < 1$ are found by minimizing the Euclidean action S_eff given in equation (19). Here we merely describe a single-harmonic limit $k_1 \to 0$ of solution equation (27). In this limit the expression for $P_0(\tau)$ in equation (27) turns into:

$$\tilde{\eta} P_0(\tau) \approx 2\pi nk_\mathrm{B}Tk_1 \sin(2\pi nk_\mathrm{B}T\tau). \tag{28}$$

Simultaneously, it was shown in [30] that the solution in the form of equation (27) leads to the following spectrum of the Floquet eigenvalues $\alpha_i(P)$ found from equation (20):

$$\alpha_i = 2k_\mathrm{B}T\tilde{\Delta}_i \left(\frac{1 - k^2 + \tilde{\Delta}_i^2}{1 + \tilde{\Delta}_i^2}\right)^{1/2} n\Pi\left(\frac{k^2}{1 + \tilde{\Delta}_i^2}, k\right). \tag{29}$$

Here $\Pi(m, k)$ is an elliptic integral of the third kind, and besides:

$$\tilde{\Delta}_i \equiv \frac{\Delta_i}{2nk_\mathrm{B}TK(k)}, \qquad k = 2\sqrt{k_1}/(1 + k_1); \qquad k'^2 = 1 - k^2. \tag{30}$$

The latter relation between parameters k_1, k is known as a Landen transformation [31]. The Jacobi function $M(\tau) = M(\tau + 1/nk_\mathrm{B}T)$ from equation (27) turns the generic self-consistency equation (22) into an algebraic equation for parameters k, n [29]:

$$\sum_i \left[\tanh\frac{\alpha_i}{2k_\mathrm{B}T}\right] \frac{\tilde{\Delta}_i}{\{(\tilde{\Delta}_i^2 + 1)(\tilde{\Delta}_i^2 + k'^2)\}^{1/2}} = \frac{1}{m\tilde{\eta}^2}. \tag{31}$$

It is not hard to see that in the limit $k' \to 0$ equation (31) transforms into the classical mean-field self-consistency equation (24), while in the limit $k' \to 1$ the self-consistency equation (31), as follows from equations (29) and (30), turns into the equation:

$$\sum_i \left[\tanh\frac{\Delta_i}{2k_\mathrm{B}T}\right] \frac{\Delta_i}{\Delta_i^2 + (\pi nk_\mathrm{B}T)^2} = \frac{1}{m\tilde{\eta}^2}. \tag{32}$$

Now making comparison with the equation (24) for the classical photon condensate P_0 it is possible to conclude that in the case of a narrow distribution of energy differences of the two-level systems $\Delta_i \approx \Delta_0$ and strong coupling constant to the electromagnetic field: $\Delta_0 < m\tilde{\eta}^2 N$, the quantum ordered phase (QOP) of the photon field occurs below the temperature:

$$T_n^\mathrm{QOP} \propto \left[\frac{\Delta_0^2 m\tilde{\eta}^2 N}{\pi^2}\right]^{1/3} \frac{1}{k_\mathrm{B}}. \tag{33}$$

Since the amplitude of $P_0(\tau)$ is proportional to $4nk_\mathrm{B}KTk_1$ in accordance with equation (27), this result suggests that the number of photons condensed into the quantum ordered phase is $\propto N^{1/3}$, where N is the number of two-level systems. Hence, this dependence indicates the presence of the coherent state also in the quantum ordered phase, but with less strong entangling than in the classical photon condensate described by equation (24). In the opposite case of a wide distribution of Δ_i and weak coupling constant $\Delta_0 > m\tilde{\eta}^2 N$ we find a transition temperature similar to the classical mean-field case equation (26):

$$T_w^\mathrm{QOP} = \frac{\Delta_0}{\pi} \mathrm{e}^{-\frac{\Delta_0}{m\tilde{\eta}^2 N}}. \tag{34}$$

In both cases, when temperature drops well below $T_{n,w}^\mathrm{QOP}$ the increase of the integer number $n \propto 1/T$ of the oscillations of the quantum order parameter $P(\tau)$ along the imaginary time-axis τ within the Euclidean space temporal interval

$[0, 1/T]$ keeps the QOP amplitude $4nk_BKTk_1$ finite and non-vanishing up to the $T = 0$ K state according to equation (27). Thus, we can conclude that the quantum phase transition in the photon states is the first-order type of phase transition.

5. Discussion and conclusions

In section 4 we obtained classical and quantum second-order phase transitions in the states of photons that can emerge in a resonant cavity strongly interacting with an array of two-level systems. For the classical phase transition we obtain that at low temperatures, $T < T^\star$, the incoherent photon state (with $P_0 = 0$) becomes unstable. Two novel photon states characterized by non-zero values of the momentum of photon field $\pm P_0(T)$ can appear at low temperatures. These photon states are the coherent photon states well known in quantum optics [32]. The average number of photons in these states $\bar{n} = P_0^2/(2m\hbar\omega_0)$. The temperature dependence of $P_0(T)$ is determined by equation (24) and is shown in figure 2 (red solid line). The value of $T^\star$ is determined by equations (25) and (26). The critical temperature $T^\star$ depends strongly on the distribution of energy differences Δ_i, the number of qubits N and the strength of interaction $\tilde{\eta}$. For example, we obtain an expression for $T_n^\star$ as $T_n^\star = \frac{\ell E_J^2}{4C_0 k_B c^2 \Phi_0^2}[(\delta\varphi_i)^2\eta^2 N/\beta_L^2] = T_{\rm eff}\frac{(\delta\varphi_i)^2\eta^2 N}{\beta_L^2}$. For typical parameters $\ell = 1$ cm, $C_0 = 8.8 \times 10^{-12}$ F m^{-1} and $E_J = 10^{-2}$ eV the effective temperature is $T_{\rm eff} \simeq 50$ K. Thus, for reasonable values of parameters, $T^\star$ varies from 100 mK up to 50 K. This phase transition is similar to a well-known metal–ferromagnet transition and the momentum of the photon field $\pm P_0(T)$ corresponds to the 'Weiss effective magnetic field' in a theory of ferromagnetic materials [25].

Notice here that these two coherent photon states, corresponding to the $+P_0(T)$ and $-P_0(T)$ values, are the degenerate ones. In this case quantum beating between these two states can be observed in the system. The quantum beating between two photon states results in a splitting of resonant frequencies of the cavity, i.e. $\omega_{\rm res} = \omega_0 \pm \Delta\omega$. The splitting $\Delta\omega$ is obtained as follows: the two stable photon states are separated by the effective potential barrier $\Delta U \simeq mN^2\tilde{\eta}^2$ (for $T = 0$) and, therefore, $\Delta\omega \simeq \omega_0 \exp[-\Delta U/(\hbar\omega_0)]$. The effect of the splitting of resonant frequencies is a consequence of the degeneracy of the photon states. Such a degeneracy can be lifted by application of an external magnetic field allowing one to realize a non-symmetric double-well potential for RF SQUIDs, i.e. as $\epsilon_i \neq 0$. In this case a single coherent state of photons with $P_0 \propto \sum_i \epsilon_i$ emerges in the cavity.

We obtained also that different metastable states of photons can be obtained in this system. In these states there is no net classical photon condensate, but there exists a macroscopic quantum condensate (with amplitude of the photon momentum $P \propto N^{1/3}$, where N is the number of two-level systems) that has a zero mean value of the electromagnetic field. These states appear as a result of a first-order phase transition.

In conclusion we have shown that superconducting quantum metamaterials can support the diverse non-classical photon states. As a particular example of such a metamaterial we considered an array of RF SQUIDs incorporated in a low-dissipative resonant cavity. We mapped this system onto a set of two-level systems (qubits) strongly interacting with photons of the cavity. By making use of a complete quantum-mechanical description of such a system we found that at high temperatures, $T > T^\star$, the incoherent chaotic state of photons is a stable one. At low temperatures, $T < T^\star$, a large number of different photon states emerges in the cavity. These photon states appear as a result of specific classical (the second-order type) or quantum (the first-order type) phase transitions. The physical origin of such phase transitions is as follows: a strong interaction of EF with two-level systems leads to the effective enhancement of the difference in energy levels of the qubits which, in turn, changes the EF in the cavity. The order parameter of phase transitions is the τ-dependent momentum of photon field $P_0(\tau)$. In the case of the classical phase transition, as $P_0(\tau) = \text{const}$ the coherent photon states and the quantum superposition of two coherent photon states can be obtained in the cavity. In the case of quantum phase transitions, the different metastable photon states characterized by a complex dependence of $P(\tau)$ (see equation (27)) can also be realized.

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